

Surface bundles over surfaces: new inequalities between signature, simplicial volume and Euler characteristic

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Abstract

We present three new inequalities tying the signature, the simplicial volume and the Euler characteristic of surface bundles over surfaces. Two of them are true for any surface bundle, while the third holds on a specific family of surface bundles, namely the ones that arise through a ramified covering. These are the main known examples of bundles with non-zero signature.

1 Introduction

Surface bundles over surfaces form an interesting family of 4-manifolds that gave rise to several questions: for example, do such manifolds with non-zero signature exist? If yes, which values does the signature take? What are the minimal base and fibre genera required to achieve a given signature? The relation and inequalities between signature and Euler characteristic of surface bundles have been widely studied, notably by Bryan, Catanese, Donagi, Endo, Korkmaz, Kotschick, Ozbagci, Rollenske, Stipsicz in [BD, BDS, CR, EKKOS, Ko].

In the present note we add the comparison to the simplicial volume of the total space, using the tools provided by the theory of bounded cohomology. The simplicial volume can act as a bridge between the two other invariants (for the definition of simplicial volume, see Subsection 2.2).

The best known inequality between signature and Euler characteristic valid for any surface bundle E over a surface is due to Kotschick [Ko]:

$$2|\sigma(E)| \leq \chi(E).$$

We compare here the signature to the simplicial volume of such bundles and obtain:

Theorem 1. *Let E be an oriented surface bundle over a surface. Then*

$$36|\sigma(E)| \leq \|E\|.$$

Observe that this is stronger than the combination of Kotschick's inequality and the first author's lower bound $6\chi(E) \leq \|E\|$ for the simplicial volume [B2], which would only give $12|\sigma(E)| \leq \|E\|$.

The simplicial volume remains very hard to compute explicitly. In fact, the exact values are known only for hyperbolic manifolds (due to Gromov-Thurston) and for locally- $(\mathbb{H}^2 \times \mathbb{H}^2)$ manifolds, so in particular for products of surfaces [B1].

We can give a lower bound on $\|E\|$ under the form of the simplicial volume of a distinguished subsurface:

Proposition 2. *Let N be the dual of the Euler class of the tangent bundle along the fibre of E . Then*

$$||N|| \leq \frac{1}{3}||E||.$$

The tangent bundle along the fibre will be defined in Subsection 2.3. Observe that the dual of the Euler class can indeed be represented by a subsurface of E , hence once we know its minimal genus we will be able to compute its simplicial volume. Unfortunately for now the known lower bounds on $||N||$ do not produce better inequalities for $||E||$ than the already known ones.

Signatures remain, analogously to simplicial volume, quite hard to calculate for general surface bundles and are essentially only computed for bundles coming from specific constructions: differences of Lefschetz fibrations or ramified coverings. We will specialise to the latter family of examples (see Section 5 for the definition and notations) and prove:

Theorem 3. *Let E be a surface bundle as in Section 5.2. Then*

$$||E|| \geq 6\chi(E) + 6|\chi(\Sigma')|(d-1),$$

where Σ' is one of the surfaces of the ramified covering construction and d the degree of the ramified covering.

Remark that this improves the inequality $||E|| \geq 6\chi(E)$ of the first author.

In the next section we recall the definitions of the invariants under consideration and the main tools to compute them. We devote Section 3 to the proof of Theorem 1 and Section 4 to the proof of Proposition 2. The bundles related to ramified coverings will be treated in Section 5.

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2 Definition of the invariants

In what follows we study oriented surface bundles over surfaces $F \hookrightarrow E \xrightarrow{\pi} B$, where both F and B are closed.

While the Euler characteristic does not need to be redefined, let us just recall that it is multiplicative in the base and the fibre of a bundle, that is it satisfies

$$\chi(E) = \chi(F)\chi(B).$$

In particular all the bundles with same base and fibre have the same Euler characteristic.

2.1 Signature

The signature of a closed connected oriented $4k$ -manifold M , where $k \in \mathbb{N}$, is defined as follows.

Consider the bilinear form induced by the cup product on the middle-dimensional cohomology groups:

$$\begin{aligned} \cup : H^{2k}(M, \mathbb{Z}) \times H^{2k}(M, \mathbb{Z}) &\longrightarrow H^{4k}(M, \mathbb{Z}) \cong \mathbb{Z} \\ (\alpha, \beta) &\longmapsto \alpha \cup \beta. \end{aligned}$$

As $\alpha \cup \beta = (-1)^{2k \cdot 2k} \beta \cup \alpha = \beta \cup \alpha$, the form is symmetric. Thus all its eigenvalues are real, and we can compute its signature in \mathbb{Z} as the number of positive eigenvalues $b_2^+(E)$ minus the number of negative eigenvalues $b_2^-(E)$. The 0 eigenvalues are neglected.

The signature of M , denoted by $\sigma(M)$, is the signature of the above bilinear form.

2.2 Simplicial volume

Let X be a topological space.

One can define a semi-norm on homology classes in the singular homology: let $\zeta \in H_k(X, \mathbb{R})$ be a homology class. Then

$$\|\zeta\|_1 = \inf \left\{ \sum_i |a_i| \left| \left[\sum_i a_i \sigma_i \right] = \zeta \right. \right\},$$

where $\sigma_i : \Delta^k \rightarrow X$ denotes a singular simplex of dimension k . We call this norm the l_1 -norm.

The simplicial volume of a closed oriented manifold M of dimension n is then defined as the l_1 -norm of its fundamental class $[M]$,

$$\|M\| = \inf \left\{ \sum_i |a_i| \left| [M] = \left[\sum_i a_i \sigma_i \right] \in H_n(M, \mathbb{R}) \right. \right\}.$$

This invariant was introduced by Gromov in [G], as a topological measure of the complexity of a manifold.

We will also need the norm commonly used in the theory of bounded cohomology, but which we consider on standard singular cohomology classes. Let $\beta \in H^k(X, \mathbb{R})$ be a cohomology class. The norm of β is defined as the infimum of the sup norm of all cochains representing β :

$$\|\beta\| = \inf \left\{ \|b\|_\infty \mid b \in C^k(X, \mathbb{R}), [b] = \beta \right\}.$$

Note that it is possible that $\|b\|_\infty = \infty$ for every such b and in particular that $\|\beta\| = \infty$.

We will use the following relationship between l_1 -norm and sup norm:

Proposition 4 ([BP], Proposition F.2.2). *Let $\beta \in H^k(X, \mathbb{R}), \zeta \in H_k(X, \mathbb{R})$ as above. Then*

$$\frac{|\langle \beta, \zeta \rangle|}{\|\beta\|} \leq \|\zeta\|_1.$$

If M is an oriented compact n -dimensional manifold and $\beta \in H^n(M, \mathbb{R})$ is a cohomology class of degree n , then

$$\frac{|\langle \beta, [M] \rangle|}{\|\beta\|} = \|M\|.$$

2.3 The Euler class

The space E admits a so-called tangent bundle along the fibre, namely

$$T\pi = \{v \in TE \mid \pi_*(v) = 0\}.$$

As an oriented vector bundle, it has an Euler class. We call it the Euler class of the bundle E and denote it by $e \in H^2(E, \mathbb{Z})$. Its Poincaré dual $e \cap [E] \in H_2(E, \mathbb{Z})$ will be denoted by $[N]$. Note that $[N]$, as a degree 2 homology class in a 4-manifold, is representable by a subsurface of E .

Passing through the isomorphism $H^2(E, \mathbb{Z}) \cong H^2(\pi_1(E), \mathbb{Z})$, true for aspherical E , the Euler class e can be represented by the cocycle $\frac{1}{2}\rho^*(Or)$, where Or denotes the orientation cocycle defined on the unit circle and ρ a certain homomorphism $\pi_1(E) \rightarrow \text{Homeo}^+(S^1)$ (see [B2, p. 85] and [M1, Propositions 4.1 and 4.3]). The definition of the orientation cocycle on the circle S^1 is

$$Or : (S^1)^3 \longrightarrow \mathbb{Z}$$

$$(x, y, z) \longmapsto \begin{cases} 1 & \text{if } x, y, z \text{ are distinct and positively oriented,} \\ 0 & \text{if two points among } x, y, z \text{ coincide,} \\ -1 & \text{if } x, y, z \text{ are distinct and negatively oriented.} \end{cases}$$

It is alternating and its norm as a cocycle is obviously 1. Therefore the class e has an alternating representative and its norm is $\|e\| \leq \frac{1}{2}$.

The signature of a surface bundle E over a surface as above can be computed using the following proposition:

Proposition 5 (See [M2], Proposition 4.11). *Let $p : E \rightarrow B$ be an oriented surface bundle over a closed oriented surface B . Then*

$$3\sigma(E) = \langle e \cup e, [E] \rangle.$$

3 Proof of Theorem 1

Let us restate the result:

Theorem 1. *Let $F \hookrightarrow E \rightarrow B$ be an oriented surface bundle over a surface, with closed oriented base and fibre. Then*

$$\|E\| \geq 36|\sigma(E)|.$$

Proof. By Proposition 5, we have

$$\langle e \cup e, [E] \rangle = 3\sigma(E).$$

On the other hand,

$$|\langle e \cup e, [E] \rangle| = \|e \cup e\| \cdot \|E\| \leq \frac{1}{12}\|E\|,$$

where we have used that $\|e \cup e\| \leq \frac{1}{12}$ (see [B2, p. 85], and [B3, p. 337 and Lemma 8]).

Hence $\|E\| \geq 12 \cdot 3|\sigma(E)| = 36|\sigma(E)|$. \square

Remark 6. *In 1998, Kotschick proved the following theorem:*

Theorem 7 ([Ko], Theorem 2). *Let E be an aspherical surface bundle over a surface. Then*

$$2|\sigma(E)| \leq \chi(E).$$

The first author then obtained the following result:

Theorem 8 ([B2], Corollary 1.3). *Let X be a fibre bundle with fibre F over a closed oriented manifold B . If $\dim(X) \leq 4$, then*

$$\|X\| \geq \|F \times B\|.$$

In the case of F and B being closed surfaces, a preceding result of the first author gives a more familiar look to the right-hand side:

Theorem 9 ([B1], Corollary 3). *Let F and B be closed oriented surfaces. Then*

$$||F \times B|| = \frac{3}{2} ||F|| \cdot ||B||.$$

Remember that $||F|| = 2|\chi(F)|$ for any closed oriented hyperbolic surface F , and that $\chi(E) = \chi(F)\chi(B)$ for any F -bundle over B . Putting everything together, we obtain:

$$||E|| \geq \frac{3}{2} \cdot 2|\chi(F)| \cdot 2|\chi(B)| = 6\chi(E) \geq 12|\sigma(E)|,$$

which is weaker than the inequality of Theorem 1.

4 Proof of Proposition 2

Recall the alternation of a chain:

Definition 10. *Let X be a geodesic space, let $\sigma = [v_0, \dots, v_n] \in C_n(X, \mathbb{Q})$ be a geodesic simplex in X given by its vertices. Define $\text{Alt}(\sigma)$ by*

$$\text{Alt}(\sigma) = \frac{1}{(n+1)!} \sum_{\tau \in \text{Sym}(n+1)} \text{sign}(\tau) [v_{\tau(0)}, \dots, v_{\tau(n)}] \in C_n(X, \mathbb{Q}).$$

Denote by σ^τ the simplex $[v_{\tau(0)}, \dots, v_{\tau(n)}]$ obtained from σ by permuting the vertices of σ by τ .

The definition is extended by linearity on the whole group $C_n(X, \mathbb{Q})$.

Remark 11. *It is known that a cycle and its alternation define the same class in $H_n(X, \mathbb{Q})$, that is $[z] = [\text{Alt}(z)] \in H_n(X, \mathbb{Q})$.*

Remark 12. *Using the triangle inequality, one readily sees that for any $z \in C_n(X, \mathbb{Q})$,*

$$||\text{Alt}(z)||_1 \leq ||z||_1.$$

Proposition 2. *Let $F \hookrightarrow E \rightarrow B$, where both F and B are closed, be a surface bundle over a surface, and N be the Poincaré dual of its Euler class as defined in Subsection 2.3. Then*

$$||N|| \leq \frac{1}{3} ||E||.$$

Proof. Choose a fundamental cycle $\sum_{i=1}^k a_i \sigma_i$ representing $[E]$. By definition, $[N] = e \cap [E]$.

Note that as $||e|| \leq \frac{1}{2}$, we already have by Proposition 4 that $2||N|| \leq ||E||$. With some more care we will get the better inequality of our proposition. The proof follows the very same idea of the computation in [B3, p. 337].

By Remark 11, we have

$$[N] = e \cap [\text{Alt}(E)] = \sum_{i=1}^k a_i \frac{1}{5!} \sum_{\tau \in \text{Sym}(5)} \text{sign}(\tau) e(\sigma_i^\tau) [[\sigma_i^\tau]].$$

Fix $i \in \{1, \dots, k\}$ and denote σ_i by $[v_0, \dots, v_4]$.

We start by showing that the expression $\frac{1}{5!} \sum_{\tau \in \text{Sym}(5)} \text{sign}(\tau) e(\sigma_i^\tau) [[\sigma_i^\tau]]$ can be greatly simplified, using the fact that the class e is representable by an alternating cocycle.

First we show that

$$T(j) := \frac{1}{4} \sum_{\tau \in \text{Sym}(5), \tau(2)=j} \text{sign}(\tau) e([v_{\tau(0)}, v_{\tau(1)}, v_{\tau(2)}]) [v_{\tau(2)}, v_{\tau(3)}, v_{\tau(4)}]$$

is independent of j , $0 \leq j \leq 4$.

By definition, $T(0)$ is

$$\begin{aligned} & e([v_0, v_3, v_4]) [v_0, v_1, v_2] - e([v_0, v_2, v_4]) [v_0, v_1, v_3] \\ & + e([v_0, v_2, v_3]) [v_0, v_1, v_4] + e([v_0, v_1, v_2]) [v_0, v_3, v_4] \\ & - e([v_0, v_1, v_3]) [v_0, v_2, v_4] + e([v_0, v_1, v_4]) [v_0, v_2, v_3], \end{aligned}$$

where we have used that e is alternating and that changing the sign in front of a simplex changes its orientation.

From the cycle and cocycle relations, for $i, j \in \{2, 3, 4\}$, we have

$$e([v_0, v_i, v_j]) = e([v_1, v_i, v_j]) + e([v_0, v_1, v_j]) - e([v_0, v_1, v_i])$$

and

$$[v_0, v_i, v_j] = [v_1, v_i, v_j] + [v_0, v_1, v_j] - [v_0, v_1, v_i].$$

The above expression thus becomes

$$\begin{aligned} & (e([v_1, v_3, v_4]) + e([v_0, v_1, v_4]) - e([v_0, v_1, v_3])) [v_0, v_1, v_2] \\ & - (e([v_1, v_2, v_4]) + e([v_0, v_1, v_4]) - e([v_0, v_1, v_2])) [v_0, v_1, v_3] \\ & + (e([v_1, v_2, v_3]) + e([v_0, v_1, v_3]) - e([v_0, v_1, v_2])) [v_0, v_1, v_4] \\ & + e([v_0, v_1, v_2]) ([v_1, v_3, v_4] + [v_0, v_1, v_4] - [v_0, v_1, v_3]) \\ & - e([v_0, v_1, v_3]) ([v_1, v_2, v_4] + [v_0, v_1, v_4] - [v_0, v_1, v_2]) \\ & + e([v_0, v_1, v_4]) ([v_1, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]). \end{aligned}$$

Note that all the terms of the form $e([v_0, v_1, v_i])[v_0, v_1, v_j]$ for $i, j \in \{2, 3, 4\}$ cancel out two by two, and that what remains left is exactly $T(1)$. Hence $T(0) = T(1)$. Applying the cyclic permutation $(0, 1, 2, 3, 4)$ and its powers to the indices in the proof of the latter equality, we see that $T(1) = T(2) = T(3) = T(4)$.

Hence $\frac{1}{5!} \sum_{\tau \in \text{Sym}(5)} \text{sign}(\tau) e(\sigma_i^\tau) [\sigma_i^\tau]$ is equal to

$$\frac{1}{30} (T(0) + T(1) + T(2) + T(3) + T(4)) = \frac{1}{6} T(0).$$

From this follows the equality

$$\begin{aligned} [N] &= \frac{1}{6} \sum_{i=1}^k a_i (e([v_0^i, v_3^i, v_4^i]) [v_0^i, v_1^i, v_2^i] - e([v_0^i, v_2^i, v_4^i]) [v_0^i, v_1^i, v_3^i] \\ & + e([v_0^i, v_2^i, v_3^i]) [v_0^i, v_1^i, v_4^i] + e([v_0^i, v_1^i, v_2^i]) [v_0^i, v_3^i, v_4^i] \\ & - e([v_0^i, v_1^i, v_3^i]) [v_0^i, v_2^i, v_4^i] + e([v_0^i, v_1^i, v_4^i]) [v_0^i, v_2^i, v_3^i]), \end{aligned}$$

where v_j^i , $0 \leq j \leq 4$, denote the vertices of the simplex σ_i , $1 \leq i \leq k$.

Let us now consider each i -th summand separately. There are 24 relative cyclic orderings of the vertices v_0^i, \dots, v_4^i , corresponding to the elements of $\text{Sym}(4)$, inducing different sets of values for $e(v_0^i, v_j^i, v_k^i)$, for $j, k \in \{1, 2, 3, 4\}$. For each of these orderings, we will find a simplification of the expression $(*)$

$$\begin{aligned} (*) &= e([v_0^i, v_3^i, v_4^i]) [v_0^i, v_1^i, v_2^i] - e([v_0^i, v_2^i, v_4^i]) [v_0^i, v_1^i, v_3^i] \\ & + e([v_0^i, v_2^i, v_3^i]) [v_0^i, v_1^i, v_4^i] + e([v_0^i, v_1^i, v_2^i]) [v_0^i, v_3^i, v_4^i] \\ & - e([v_0^i, v_1^i, v_3^i]) [v_0^i, v_2^i, v_4^i] + e([v_0^i, v_1^i, v_4^i]) [v_0^i, v_2^i, v_3^i]. \end{aligned}$$

We will use three cycle relations for subsimplices of $[v_0^i, \dots, v_4^i]$:

$$[v_2^i, v_3^i, v_4^i] = [v_0^i, v_3^i, v_4^i] - [v_0^i, v_2^i, v_4^i] + [v_0^i, v_2^i, v_3^i],$$

$$[v_1^i, v_3^i, v_4^i] = [v_0^i, v_1^i, v_3^i] - [v_0^i, v_1^i, v_4^i] + [v_0^i, v_3^i, v_4^i],$$

and

$$[v_1^i, v_2^i, v_3^i] = [v_0^i, v_1^i, v_2^i] - [v_0^i, v_1^i, v_3^i] + [v_0^i, v_2^i, v_3^i].$$

The first will be used with the cyclic orderings induced by the identity and the following permutations:

$$(1, 4), (2, 3), (2, 4), (3, 4), (1, 4)(2, 3), (1, 4, 2), (1, 4, 3), (2, 3, 4), (2, 4, 3), (1, 4, 3, 2), (1, 4, 2, 3);$$

the second will take care of the orderings induced by

$$(1, 2), (1, 2)(3, 4), (1, 3)(2, 4), (1, 3, 2), (1, 2, 4), (1, 3, 2, 4), (1, 2, 4, 3), (1, 3, 4, 2);$$

finally the third will be applied to the orderings coming from:

$$(1, 3), (1, 2, 3), (1, 3, 4), (1, 2, 3, 4).$$

The important observation is that for orderings of the first kind, the values of

$$e([v_0^i, v_1^i, v_2^i]), e([v_0^i, v_1^i, v_3^i]), e([v_0^i, v_1^i, v_4^i])$$

are all 1 (or all -1), allowing to replace

$$e([v_0^i, v_1^i, v_2^i]) [v_0^i, v_3^i, v_4^i] - e([v_0^i, v_1^i, v_3^i]) [v_0^i, v_2^i, v_4^i] + e([v_0^i, v_1^i, v_4^i]) [v_0^i, v_2^i, v_3^i]$$

by $[v_2^i, v_3^i, v_4^i]$ (or by $-[v_2^i, v_3^i, v_4^i]$).

For orderings of the second kind, the relevant values are

$$e([v_0^i, v_1^i, v_2^i]) = 1, e([v_0^i, v_2^i, v_4^i]) = -1, e([v_0^i, v_2^i, v_3^i]) = -1,$$

(or the opposites), allowing to replace

$$e([v_0^i, v_1^i, v_2^i]) [v_0^i, v_3^i, v_4^i] - e([v_0^i, v_2^i, v_4^i]) [v_0^i, v_1^i, v_3^i] + e([v_0^i, v_2^i, v_3^i]) [v_0^i, v_1^i, v_4^i]$$

by $[v_1^i, v_3^i, v_4^i]$ (or by $-[v_1^i, v_3^i, v_4^i]$).

Finally for orderings of the third kind, we consider

$$e([v_0^i, v_3^i, v_4^i]), e([v_0^i, v_2^i, v_4^i]), e([v_0^i, v_1^i, v_4^i]),$$

which are all 1 (or all -1), allowing to replace

$$e([v_0^i, v_3^i, v_4^i]) [v_0^i, v_1^i, v_2^i] - e([v_0^i, v_2^i, v_4^i]) [v_0^i, v_1^i, v_3^i] + e([v_0^i, v_1^i, v_4^i]) [v_0^i, v_2^i, v_3^i]$$

by $[v_1^i, v_2^i, v_3^i]$ (or by $-[v_1^i, v_2^i, v_3^i]$).

Hence whatever the cyclic ordering of the vertices of σ_i is, we can replace the 6 terms of the initial expression $(*)$ by a 4-terms expression.

We thus have reduced the number of simplices appearing in the expression of N by a factor of $\frac{4}{6} = \frac{2}{3}$ with respect to the number of simplices appearing in $Alt(E)$. Recall once more that the norm $\|e\|$ is at most $\frac{1}{2}$.

So we get the following inequality:

$$\|N\| \leq \frac{1}{2} \cdot \frac{2}{3} \|Alt(E)\| \leq \frac{1}{3} \|E\|,$$

proving the proposition. \square

5 Ramified coverings

In this section we present the method for constructing surface bundles with non-zero signature using ramified coverings and then study the simplicial volume of the total space of such bundles.

5.1 Construction of surface bundles using ramified coverings

The first examples of surface bundles over surfaces with non-zero signature were constructed independently by Kodaira [K] in 1967 and Atiyah [A] in 1969 with a method relying on ramified coverings. We outline this method here, following its exposition in [M2, Paragraph 4.3.3].

First choose a closed oriented surface $B_1 = \Sigma_{g_1}$, with $g_1 \geq 2$. Then take a d -fold cyclic covering $\rho_1 : B_2 \rightarrow B_1$ of B_1 , and let σ be a generator of its covering transformation group $\mathbb{Z}/d\mathbb{Z}$.

Remark 13. *This implies that σ^i is fixed point free, for $1 \leq i \leq d-1$.*

Denote by g_2 the genus of B_2 . We have $2 - 2g_2 = d(2 - 2g_1)$. Consider the following homomorphisms:

$$\pi_1(B_2) \longrightarrow H_1(B_2, \mathbb{Z}) \cong \pi_1(B_2)^{ab} \longrightarrow H_1(B_2, \mathbb{Z}/d\mathbb{Z}) \cong (\mathbb{Z}/d\mathbb{Z})^{2g_2}.$$

Their composition is surjective and its kernel is a normal subgroup of finite index of $\pi_1(B_2)$. As such, it defines a finite regular covering $\rho_2 : B_3 \rightarrow B_2$. We have a map $\sigma^i \circ \rho_2 : B_3 \rightarrow B_2$ for each $1 \leq i \leq d$. We can then consider the graph of $\sigma^i \circ \rho_2$ in $B_3 \times B_2$ for each i : it defines a submanifold $\Gamma_{\sigma^i \rho_2}$.

Remark 14. *The fact that σ^i is fixed point free for all $1 \leq i \leq d-1$ ensures that the graphs $\Gamma_{\sigma^i \rho_2}, \Gamma_{\sigma^j \rho_2}$ are disjoint whenever $i \neq j$.*

Take the topological sum $\Gamma_{\sigma \rho_2} + \dots + \Gamma_{\sigma^d \rho_2}$ of these submanifolds and denote it by D . It is of codimension 2 in $B_3 \times B_2$, therefore it defines a class $[D] \in H_2(B_3 \times B_2, \mathbb{Z})$.

The following proposition will be used soon:

Proposition 15 ([M2], Proposition 4.10). *Let B be a closed oriented C^∞ manifold and let $D \subset B$ be an oriented submanifold of codimension 2. Suppose that, for some $d \in \mathbb{Z}_{\geq 0}$, the homology class $[D] \in H_{n-2}(B, \mathbb{Z})$ determined by D is divisible by d in $H_{n-2}(B, \mathbb{Z})$. Then there exists a d -fold cyclic ramified covering $\tilde{B} \rightarrow B$ ramified along D .*

One can prove that the class $[D]$ defined above is divisible by d in $H_2(B_3 \times B_2, \mathbb{Z})$. Thus using Proposition 15, one obtains a ramified covering $f : E \rightarrow B_3 \times B_2$ of degree d ramified along D .

Finally we obtain a surface bundle $E \rightarrow B_3$ as the composition $E \xrightarrow{f} B_3 \times B_2 \rightarrow B_3$, where $B_3 \times B_2 \rightarrow B_3$ is the canonical projection to the first factor. The fibre of E is $f^{-1}(B_2)$. One can explicitly compute the signature of E and see that it is non-zero.

For this, one more result is used, giving relations between the Euler class of E and the one of $B_3 \times B_2$.

Proposition 16 ([M2], Proposition 4.12). *Let $\pi : E \rightarrow B$ and $\tilde{\pi} : \tilde{E} \rightarrow B$ be two surface bundles over the same base space B . Suppose that there is a map $f : \tilde{E} \rightarrow E$ between the total spaces which is a d -fold cyclic ramified covering ramified along an oriented submanifold $D \subset E$ of codimension 2, and that f is a bundle map (i. e. $\pi \circ f = \tilde{\pi}$). Suppose also that D intersects each fibre of π transversely at exactly d points, and write $\tilde{D} = f^{-1}(D)$. Then:*

1. $f^*(\nu) = d\tilde{\nu}$;
2. $\tilde{e} = f^*(e - (1 - 1/d)\nu)$,

where ν , respectively $\tilde{\nu}$, represents the Poincaré dual of the homology class of D , respectively \tilde{D} , and e , respectively \tilde{e} , denotes the Euler class of π , respectively $\tilde{\pi}$.

All the assumptions of Proposition 16 are satisfied by $E \xrightarrow{f} B_3 \times B_2$.

5.2 Simplicial volume of such bundles

As observed in Remark 6, the 2009 results of the first author show that

$$||E|| \geq 6\chi(E)$$

for any surface bundle E over a surface.

Now if we consider surface bundles over surfaces coming from the ramified covering construction explained in the previous subsection, we can enhance this inequality.

Consider the following diagram that represents the aforementioned construction.

$$\begin{array}{ccc} E & \xrightarrow{f} & \Sigma' \times \Sigma \\ & \swarrow p' & \searrow p \\ \Sigma' & & \Sigma \end{array}$$

The maps p and p' are the natural projections. The map f is a cyclically ramified covering of degree d of $\Sigma' \times \Sigma$, ramified along the codimension 2 submanifold $D \subset \Sigma' \times \Sigma$ defined before, and Σ' is a d' -fold covering of Σ . The intersection (both algebraic and geometric) $D \cap \Sigma'$ in $\Sigma' \times \Sigma$ consists of $d'd$ points while the intersection $D \cap \Sigma$ consists of d points.

Remark 17. In order to avoid heavy notation, by $\Sigma \subset \Sigma' \times \Sigma$ we mean the choice of a subsurface $\{x\}' \times \Sigma$. We mean the same when we write $[\Sigma] \in H^2(\Sigma' \times \Sigma, \mathbb{Z})$.

The crucial remark, already made by Bryan, Donagi and Stipsicz in [BDS] and LeBrun in [L], is that E admits (at least) two different bundle structures: namely the compositions $p \circ f$ and $p' \circ f$ are the bundle projections of the surface bundles $E \rightarrow \Sigma$ and $E \rightarrow \Sigma'$ with fibres $f^{-1}(\Sigma)$ and $f^{-1}(\Sigma')$ respectively.

Theorem 3. For the total space E of a bundle as above, we have

$$||E|| \geq 6d|\chi(\Sigma')|(|\chi(\Sigma)| + (d-1)(1+1/d)).$$

Proof. Denote by $e' = \chi(\Sigma')[\Sigma]^*$ the Euler class of the product bundle $\Sigma' \times \Sigma \rightarrow \Sigma$, and by $e = \chi(\Sigma)[\Sigma']^*$ the Euler class of the product bundle $\Sigma' \times \Sigma \rightarrow \Sigma'$, both in $H^2(\Sigma' \times \Sigma, \mathbb{Z})$. The notation $[A]^*$ stands for the Poincaré dual of the homology class $[A]$.

By Proposition 16, the Euler class of the bundle $E \rightarrow \Sigma$ is $e'_E = f^*(e' - (1 - 1/d)[D]^*)$ and the one of the bundle $E \rightarrow \Sigma'$ is $e_E = f^*(e - (1 - 1/d)[D]^*)$, both in $H^2(E, \mathbb{Z})$.

Compute:

$$\begin{aligned}
\langle e'_E \cup e_E, [E] \rangle &= \left\langle f^* \left(e' - (1 - \frac{1}{d})[D]^* \right) \cup f^* \left(e - (1 - \frac{1}{d})[D]^* \right), [E] \right\rangle \\
&= d \left\langle \left(e' - (1 - \frac{1}{d})[D]^* \right) \cup \left(e - (1 - \frac{1}{d})[D]^* \right), [\Sigma' \times \Sigma] \right\rangle \\
&= d \left\langle e' - (1 - 1/d)[D]^*, \chi(\Sigma)[\Sigma'] - (1 - \frac{1}{d})[D] \right\rangle \\
&= d \left(\chi(\Sigma')\chi(\Sigma) - \chi(\Sigma')(1 - \frac{1}{d})d \right. \\
&\quad \left. - (1 - \frac{1}{d})\chi(\Sigma)dd' + (1 - \frac{1}{d})^2 dd' \chi(\Sigma) \right) \\
&= d \left(\chi(\Sigma')\chi(\Sigma) - \chi(\Sigma')(d-1) - \chi(\Sigma')(d-1) + (1 - \frac{1}{d})^2 d\chi(\Sigma') \right) \\
&= d \left(\chi(\Sigma')\chi(\Sigma) - 2\chi(\Sigma')(d-1) + (1 - \frac{1}{d})^2 d\chi(\Sigma') \right) \\
&= d\chi(\Sigma') \left(\chi(\Sigma) - 2(d-1) + (d-1)(1 - \frac{1}{d}) \right) \\
&= d\chi(\Sigma') \left(\chi(\Sigma) - (d-1)(1 + \frac{1}{d}) \right).
\end{aligned}$$

By Proposition 2.1 of [B2] and again because $\|e_E\| \leq \frac{1}{2}$, we have:

$$\|e'_E \cup e_E\| \leq \frac{1}{3}\|e_E\| \leq \frac{1}{6}.$$

So we obtain:

$$|\langle e'_E \cup e_E, [E] \rangle| = d|\chi(\Sigma')||\chi(\Sigma) - (d-1)(1 + 1/d)| \quad (1)$$

$$\frac{1}{6}\|E\| \geq \|e'_E \cup e_E\| \|E\| = d|\chi(\Sigma')| (|\chi(\Sigma)| + (d-1)(1 + 1/d)) \quad (2)$$

$$\|E\| \geq 6d|\chi(\Sigma')| (|\chi(\Sigma)| + (d-1)(1 + 1/d)) \quad (3)$$

□

Remark 18. This bound is better than $\|E\| \geq 6\chi(E)$, which can be rewritten as

$$\|E\| \geq 6d|\chi(\Sigma')| (|\chi(\Sigma)| + d - 1).$$

The difference is of $6|\chi(\Sigma')|(d-1)$.

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